Model parameter estimation with trace forecast likelihood

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Lancaster University Management School | Lancaster Centre for Forecasting

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Motivation

Parameters estimation is a key element of forecasting.

Poor estimation $\rightarrow$ poor forecasts.

Correct estimation leads to more accurate forecast.

It also decreases the uncertainty.

Conventional Estimation methods

The conventional estimation methods is based on MSE:

$$\text{MSE} = \frac{1}{T} \sum_{t=1}^{T} e_{t+1|t}^2 \tag{1}$$

where $e_{t+1|t} = y_{t+1} - \hat{y}_{t+1}$

MSE – "Mean Squared Error".

If the errors in the model are distributed normally, than using (1) is equivalent to maximising the following log-likelihood function (Hyndman et al., 2008):

$$\ell(\theta, \hat{\sigma}^2 | \mathbf{Y}) = -\frac{T}{2} \left(\log(2\pi e) + \log \hat{\sigma}^2 \right) \quad (2)$$

where $\hat{\sigma}^2$ is the estimated variance of residuals of the model, θ is a vector of parameters of the model.

This implies that we look at conditional distribution of one-step-ahead forecast error.

Advanced estimation methods

Sometimes the forecasting task is aligned to estimation:

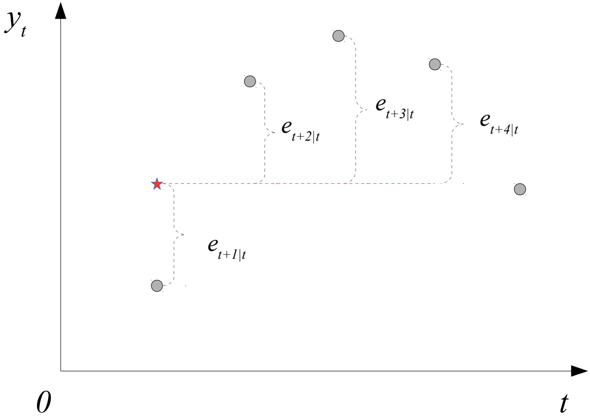
$$\text{MSE}_h = \frac{1}{T} \sum_{t=1}^{T} e^2_{t+h|t} \tag{3}$$

or:

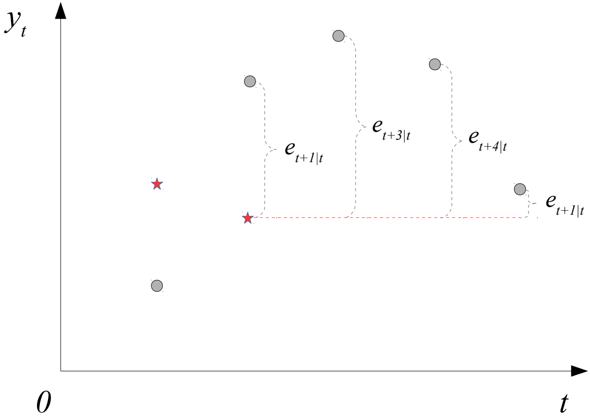
$$\text{MSTFE} = \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{h} e^2_{t+j|t} \tag{4}$$

MSTFE – "Mean Squared Trace Forecast Error".

These cost functions imply that we produce h-steps ahead forecasts from each observation:



These cost functions imply that we produce h-steps ahead forecasts from each observation:



MSE$_h$ produces robust estimates of parameters.
(**???**)
The forecast accuracy increases.
(**????**)
MSTFE is consistent.
(**?**)

BUT!
The efficiency of estimates of MSE$_h$ is low.
(**?**)
? demonstrate on a set of 170 time series that the forecast accuracy using MSE$_h$ is lower than using MSE.

Problems:

- The results are ambiguous;
- Estimates of parameters are inefficient;
- Estimates of parameters could be unstable;
- Nobody has ever explained why MSE_h and MSTFE work / don't work;

- There is no likelihood function for both MSE_h and MSTFE;
- Model selection using MSE_h and MSTFE is really tricky (??);

Why they work

It can be shown that MSE is proportional to variance of one-step-ahead error.

MSE_h is then proportional to variance of h-step-ahead error.

MSTFE is in fact the sum of MSE_h .

Using state-space approach (Snyder, 1985), variance of h-step-ahead error is:

$$\sigma_h^2 = \sigma_1^2 \left(1 + \sum_{j=1}^{h-1} c_j^2 \right) . \tag{5}$$

This means that minimising MSE$_h$ (or MSTFE) in general leads to:

1. decrease of variance of one-step-ahead error,
2. shrinkage of values of smoothing parameters towards zero,

Examples:

ETS(A,N,N): $c_j = \alpha$; $\sigma_h^2 = \sigma_1^2 \left(1 + (h-1)\alpha^2\right)$.

ETS(A,A,N): $c_j = \alpha + \beta j$; $\sigma_h^2 = \sigma_1^2 \left(1 + \sum_{j=1}^{h-1} (\alpha + \beta j)^2\right)$.

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This is root of the problem and main advantage of MSE_h and MSTFE.

If model is wrong, shrinkage allows to get rid of redundant parameters.

If model is correct, the parameters "overshrink".

The shrinkage effect becomes stronger when h increases.

Solution – Trace Forecast Likelihood (TFL)

Let's derive likelihood for multistep cost function.
We need to study multivariate distribution of errors:

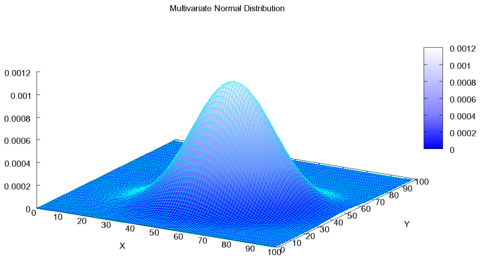

Figure: Multivariate normal distribution.

Based on multivariate normal distribution, we have (skipping derivations):

$$\ell(\theta, \hat{\mathbf{\Sigma}}|\mathbf{Y}) = -\frac{T}{2}\left(h \log(2\pi e) + \log|\hat{\mathbf{\Sigma}}|\right) \tag{6}$$

Looks similar to:

$$\ell(\theta, \hat{\sigma_1}^2|\mathbf{Y}) = -\frac{T}{2}\left(\log(2\pi e) + \log \hat{\sigma_1}^2\right) \tag{7}$$

Model selection can now be done using AIC, AICc, BIC, ...

$\boldsymbol{\Sigma}$ is covariance matrix that has the structure:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} & \dots & \sigma_{1,h} \\ \sigma_{1,2} & \sigma_2^2 & \dots & \sigma_{2,h} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1,h} & \sigma_{2,h} & \dots & \sigma_h^2 \end{pmatrix}, \qquad (8)$$

Note that $MSE_h \propto \sigma_h^2$, which makes it a special case of Σ.

And MSTFE is just the trace of Σ.

What does min of $|\mathbf{\Sigma}|$ mean?

Example with $h = 2$:

$$|\mathbf{\Sigma}| = \begin{vmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_2^2 \end{vmatrix} = \sigma_1^2 \sigma_2^2 - \sigma_{1,2}^2 \quad (9)$$

Minimising determinant of $|\mathbf{\Sigma}|$ will:

1. decrease variances,
2. increase covariances.

Covariance between i and j errors is equal to:

$$\sigma_{i,j} = \sigma_1^2 \left(c_{i,j} + \sum_{l=1}^{i-1} c_{l,j} c_{l,i} \right). \quad (10)$$

So $\log |\Sigma|$ can be rewritten as a function of variances and parameters:

$$\log |\mathbf{\Sigma}| = h \log \sigma_1^2 + \log |\mathbf{C}| \quad (11)$$

$\mathbf{C}$ depends on c_j only (thus depends on smoothing parameters).

This means that we shrink parameters...

...but shrinkage effect is weakened.

We have conducted a simulation experiment:

- Generated data using ARIMA(0,1,1).
- Applied correct model and wrong model.
- Applied MSE, MSE$_h$, MSTFE and TFL.
- With h=$\{1, 10, 20, 30, 40, 50\}$.
- Wrote down the parameters...

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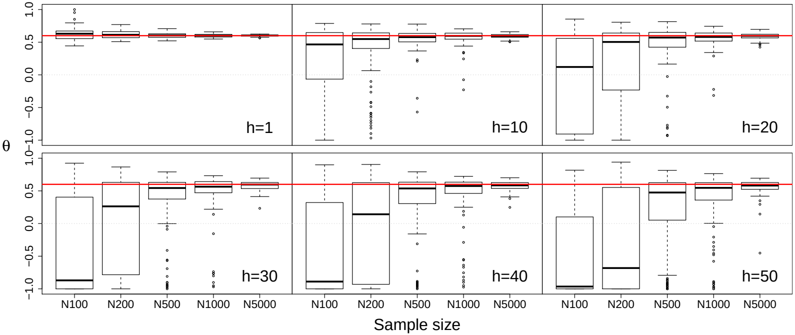
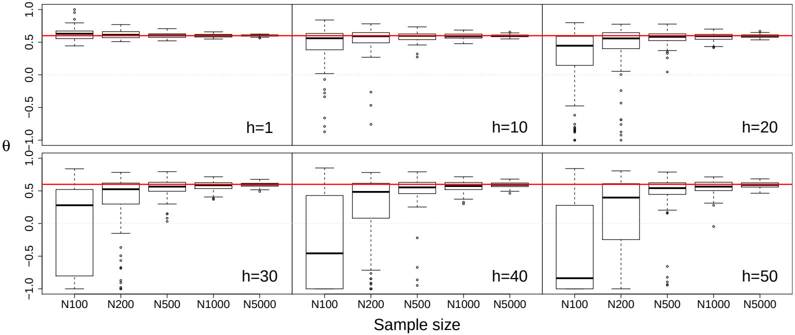
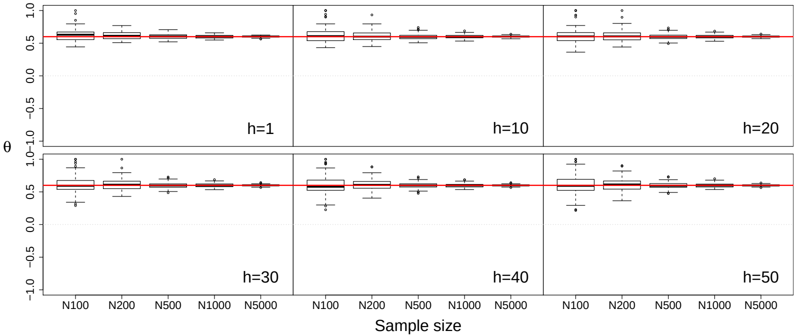
Simulations. ARIMA(0,1,1). Correct model, MSE_h



Simulations. ARIMA(0,1,1). Correct model, MSTFE



Simulations. ARIMA(0,1,1). Correct model, TFL



Conclusions

- Multiple steps ahead objective functions imply shrinkage of parameters;
- Parameters of ETS and ARIMA shrink, parameters of regressions do not;
- This gives robustness to models and help in long-term forecasting;
- Parameters may overshrink when estimated using MSE_h and MSTFE;

Conclusions

- Trace Forecast Likelihood (TFL) do not overshrink the parameters;
- TFL gives consistent, efficient and unbiased estimates of parameters;
- Model selection with TFL is possible.

TFL is brilliant in theory!

How to make it work in practice?...

Thank you for your attention!

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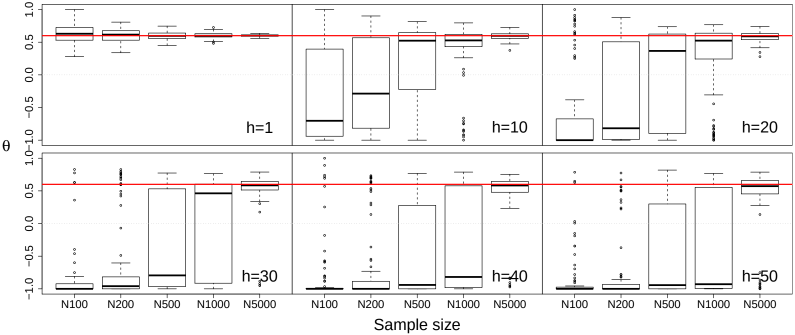
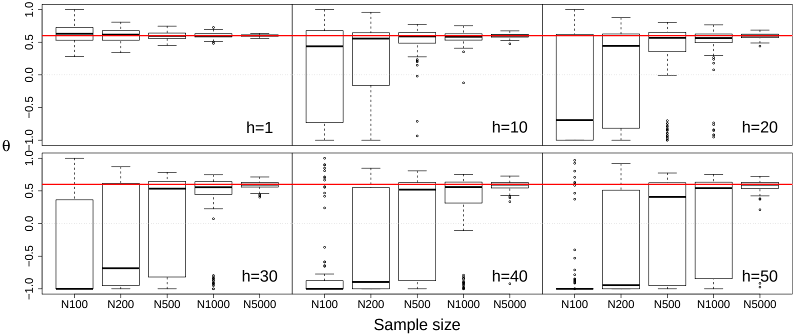
Simulations. ARIMA(0,1,1). Wrong model, MSE_h



Simulations. ARIMA(0,1,1). Wrong model, MSTFE



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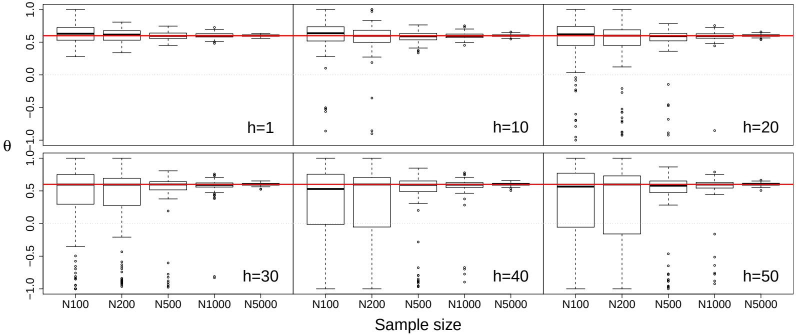

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